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Cells in Affine Weyl Groups and Tensor Categories

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In this paper we construct a tensor (or monoidal) category for any two-sided cell in a finite or affine Weyl group. The isomorphism classes of simple objects of this category correspond to the elements of the two-sided cell; the tensor product arises by a certain truncation procedure from the standard convolution of perverse sheaves on a flag manifold, which imitates geometrically the definition of the ring

correspond to the elements in the left cell intersected with its inverse (a right cell). This contains as a special case the category of equivariant perverse sheaves on an affine grassmannian, which, as a consequence of G. Lusztig (*Astérisque* **101**–**102** (1983), 208–209) is a tensor category for convolution (without truncation), closely related to the tensor category of representations of the group dual, in the sense of Langlands, to the group which gives rise to the affine grassmannian. But our construction can be specialized in other ways, and it gives a (conjectural) realization of the tensor category of representations of the reductive quotient of the centralizer of any unipotent element of the Langlands dual group. © 1997 Academic Press

1. CONVOLUTION

1.1. Throughout this paper we will use the notation of [BBD]. (Examples: ${}^p\tau_{\geqslant k}$, ${}^pH$ denote truncation and perverse cohomology sheaf, respectively.)

Let G be a simply connected semisimple algebraic group over $\mathbf{C}$, with a fixed Borel subgroup B. (The assumption that G is simply connected is not essential; we could assume that G is just semisimple and make appropriate modifications throughout the paper.) Let $\mathscr{P}$ be the set of subgroups of G that contain B. For $P \in \mathscr{P}$, we set $X_P = G/P$. If we are also given $Q \in \mathscr{P}$, we will regard X_P as a variety with Q-action (left translation) and we denote by $\mathscr{C}_{Q,P}$ the subcategory of the bounded derived category of constructible $\mathbf{C}$-sheaves on X_P consisting of complexes that are isomorphic to a finite direct sum of shifts of simple, Q-equivariant perverse sheaves on X_P.

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Let $\mathbf{O}_{Q,P}$ be the set of Q-orbits on X_P. For each $\mathcal{O}\in\mathbf{O}_{Q,P}$, we denote by $\mathbf{I}_{\mathcal{O}}$ the simple Q-equivariant perverse sheaf on X_P defined by the subvariety $\mathcal{O}$ and the local system $\mathbf{C}$ on it. Note that the objects of $\mathscr{C}_{Q,P}$ are isomorphic to finite direct sums of complexes of the form $\mathbf{I}_{\mathcal{O}}[\delta]$ for various $\mathcal{O}\in\mathbf{O}_{Q,P}$, $\delta\in\mathbf{Z}$.

Assume now that $P, Q, R\in\mathscr{P}$ and that $A\in\mathscr{C}_{R,Q}$, $A'\in\mathscr{C}_{Q,P}$. We define $A*A'\in\mathscr{C}_{R,P}$ as follows. Consider the diagram

$$X_Q\times X_P \xleftarrow{p_1} G\times G \xrightarrow{p_2} Y \xrightarrow{p_3} X_P,$$

where Y is the quotient of $G\times G$ by the free $Q\times P$-action $(q,p)\colon(g_1,g_2)\mapsto(g_1q^{-1}, qg_2p^{-1})$; p_2 is the corresponding orbit map; $p_3(g_1,g_2)=g_1g_2P$; $p_1(g_1,g_2)=(g_1Q, g_2P)$. Then p_1 and p_2 can be regarded as principal fibrations with group $Q\times P$ and p_3 is a proper morphism. We have $p_1^*(A\boxtimes A')=p_2^*(A_1)$ for a well-defined complex of sheaves A_1 on Y. We set $A*A'=(p_3)_!A_1$. We have $A*A'\in\mathscr{C}_{R,P}$, by the decomposition theorem [BBD].

1.2. We now describe an affine analogue of the definition in 1.1. (We will refer to the case discussed in 1.1 as to the "finite case" and to the present case as to the "affine case.")

Let $\tilde{G}$ be the group of points of G over $\mathbf{C}((\varepsilon))$; here ε is an indeterminate. We define a sequence of subgroups $\tilde{G}_0\supset\tilde{G}_1\supset\tilde{G}_2\supset\cdots$ of $\tilde{G}$ as follows. $\tilde{G}_0$ is the group of points of G over $\mathbf{C}[[\varepsilon]]$; for $s\geqslant 1$, $\tilde{G}_s$ is the kernel of the canonical homomorphism of $\tilde{G}_0$ onto the group of points of G over $\mathbf{C}[[\varepsilon]]/(\varepsilon^s)$.

Let $\tilde{B}$ be the inverse image of B under the canonical homomorphism $\tilde{G}_0\to G$; we regard $\tilde{B}$ as a subgroup of $\tilde{G}$ (an Iwahori subgroup). Let $\tilde{\mathscr{P}}$ be the set of subgroups of $\tilde{G}_0$ that contain $\tilde{B}$. Then, for any $Q\in\tilde{\mathscr{P}}$, and any $s\geqslant 1$, $\tilde{G}_s$ is a normal subgroup of Q.

For any $P\in\tilde{\mathscr{P}}$ we set $X_P=\tilde{G}/P$. Let $Q\in\tilde{\mathscr{P}}$. Then Q acts on X_P by left translation. Now X_P is an increasing union of Q-stable projective varieties; for any such subvariety Z we denote by $\mathscr{C}_{Q,P;Z}$ the subcategory of the bounded derived category of constructible $\mathbf{C}$-sheaves on Z consisting of complexes that are isomorphic to a finite direct sum of shifts of simple, Q-equivariant perverse sheaves on Z. (A perverse sheaf on Z is said to be Q-equivariant if it is equivariant under the algebraic group $Q/\tilde{G}_s$ for large enough s, so that $\tilde{G}_s$ acts trivially on Z). If Z' is a Q-stable projective subvariety of X_P such that $Z\subset Z'$, then we have an obvious fully faithful embedding $\mathscr{C}_{Q,P;Z}\subset\mathscr{C}_{Q,P;Z'}$ (extension by 0). We denote by $\mathscr{C}_{Q,P}$ the category obtained as the limit of $\mathscr{C}_{Q,P;Z}$ for various Z as above, with respect to the embeddings $Z\subset Z'$ above. An object of this category is said to be a Q-equivariant perverse sheaf on X_P. It has a support, defined in an obvious way (an irreducible projective Q-invariant subvariety of X_P).

Let $\mathbf{O}_{Q,P}$ be the set of Q-orbits on X_P. For each $\mathcal{O} \in \mathbf{O}_{Q,P}$, we denote by $\mathbf{I}_{\mathcal{O}}$ the simple Q-equivariant perverse sheaf on the closure of $\mathcal{O}$ defined by the subvariety $\mathcal{O}$ and the local system $\mathbf{C}$ on it. We may regard $\mathbf{I}_{\mathcal{O}}$ as an object of $\mathscr{C}_{Q,P}$. As in 1.1, the objects of $\mathscr{C}_{Q,P}$ are isomorphic to finite direct sums of complexes of the form $\mathbf{I}_{\mathcal{O}}[\delta]$ for various $\mathcal{O} \in \mathbf{O}_{Q,P}$, $\delta \in \mathbf{Z}$.

Assume now that $P, Q, R \in \tilde{\mathscr{P}}$ and that $A \in \mathscr{C}_{R,Q}$, $A' \in \mathscr{C}_{Q,P}$. We define $A * A' \in \mathscr{C}_{R,P}$ by modifying the definition in 1.1, so that infinite dimensional spaces are avoided. We choose an R-invariant projective subvariety Z of X_Q containing the support of A and a Q-invariant projective subvariety Z' of X_P containing the support of A'. Let $\hat{Z}$ (resp. $\hat{Z}'$) be the inverse image of Z (resp. Z') under the canonical map $\tilde{G} \to X_Q$ (resp. $\tilde{G} \to X_P$). We can find an R-invariant projective subvariety Z'' of X_P such that $\hat{Z}Z' \subset Z''$. We can find $s \geqslant 1$ such that $\tilde{G}_s$ acts trivially on Z'.

Let U be the orbit space for the free action of $\tilde{G}_s \times \tilde{G}_s$ on $\hat{Z} \times \hat{Z}'$ given by $(u_1, u_2)\colon (g_1, g_2) \mapsto (g_1 u_1^{-1}, u_1 g_2 u_2^{-1})$. Let U' be the orbit space for the free action of $Q/\tilde{G}_s \times P/\tilde{G}_s$ on U given by $(q, p)\colon (g_1, g_2) \mapsto (g_1 q^{-1}, q g_2 p^{-1})$. Consider the diagram

$$Z \times Z' \xleftarrow{p_1} U \xrightarrow{p_2} U' \xrightarrow{p_3} Z'',$$

where p_2 is the orbit map; $p_3(g_1, g_2) = g_1 g_2 P$; $p_1(g_1, g_2) = (g_1 Q, g_2 P)$.

Note that U, U' are naturally algebraic varieties and both p_1 and p_2 can be regarded as principal fibrations with group $Q/\tilde{G}_s \times P/\tilde{G}_s$ (an algebraic group). Moreover, p_3 is a proper morphism. We can regard $A \in \mathscr{C}_{R,Q;Z}$, $A' \in \mathscr{C}_{Q,P;Z'}$. We have $p_1^*(A \boxtimes A') = p_2^*(A_1)$ for a well defined complex of sheaves A_1 on U'. We set $A * A' = (p_3)_! A_1$. We have $A * A' \in \mathscr{C}_{R,P;Z''}$ by the decomposition theorem [BBD]. We regard $A * A'$ as an object of $\mathscr{C}_{R,P}$; it is independent of the choices made (of Z, Z', Z'', s).

1.3. Assume that we are given S, P, Q, R in $\mathscr{P}$ (resp. $\tilde{\mathscr{P}}$) and $A \in \mathscr{C}_{R,Q}$, $A' \in \mathscr{C}_{Q,P}$, $A'' \in \mathscr{C}_{P,S}$. Then there is a canonical isomorphism $(A * A') * A'' \xrightarrow{\sim} A * (A' * A'')$. In fact, both $(A * A') * A''$, $A * (A' * A'')$ are canonically isomorphic to a third object $A * A' * A'' \in \mathscr{C}_{R,S}$ defined as follows. (We will only treat the finite case; the affine case is treated in a similar way, using the same modification as in 1.2.) Consider the diagram

$$X_Q \times X_P \times X_S \xleftarrow{p'_1} G \times G \times G \xrightarrow{p'_2} Y' \xrightarrow{p'_3} X_S,$$

where Y' is the quotient of $G \times G \times G$ by the free $Q \times P \times S$-action,

$$(q, p, s)\colon (g_1, g_2, g_3) \mapsto (g_1 q^{-1}, q g_2 p^{-1}, p g_3 s^{-1});$$

p'_2 is the corresponding orbit map; $p'_3(g_1, g_2, g_3) = g_1 g_2 g_3 S$; $p'_1(g_1, g_2, g_3) = (g_1 Q, g_2 P, g_3 S)$.

Then p'_1 and p'_2 can be regarded as principal fibrations with group $Q \times P \times S$ and p'_3 is a proper morphism. We have $p'^*_1(A \boxtimes A' \boxtimes A'') = p'^*_2(A'_1)$ for a well-defined complex of sheaves A'_1 on Y'. We set $A * A' * A'' = (p'_3)_! A'_1$. We have $A * A' * A'' \in \mathcal{C}_{R,S}$, by the decomposition theorem [BBD].

1.4. For the sake of uniformity, we now make the following change of notation: we will write $B' = B$ in the finite case and $B' = \tilde{B}$ in the affine case. We shall write $\mathcal{C}$, W, instead of $\mathcal{C}_{B', B'}$, $\mathbf{O}_{B', B'}$. In the finite case, we identity W with the Weyl group of G in the standard way. In the affine case, we identity W with the affine Weyl group of G in the standard way and we denote by W_f the standard parabolic subgroup of W corresponding to the parabolic subgroup $\tilde{G}_0$ of $\tilde{G}$; let W_* be the set of elements of W that have maximal length in their (W_f, W_f)-double coset.

The convolution operation $A * A'$ in 1.1, 1.2 has been well known since the early 1980s, when it was understood that it provides a geometric interpretation for the multiplication in the Hecke algebra. (See, for example, [LV, L2, Section 3].) Namely, if $w_1, w_2 \in W$, we have

$$\mathbf{I}_{w_1} * \mathbf{I}_{w_2} \cong \bigoplus_{w_3 \in W, \delta \in \mathbf{Z}} \mathbf{I}_{w_3}[\delta]^{\oplus n_{w_1, w_2, w_3; \delta}} \qquad (\text{in } \mathcal{C}),$$

where

$$C'_{w_1} C'_{w_2} = \sum_{w_3 \in W, \delta \in \mathbf{Z}} n_{w_1, w_2, w_3;\, \delta} v^{\delta} C'_{w_3}$$

is a product of basis elements [KL] of the finite or affine Hecke algebra. Here, $n_{w_1, w_2, w_3; \delta}$ are integers.

1.5. Similarly, if (in the affine case) we identify $W_* \leftrightarrow \mathbf{O}_{\tilde{G}_0, \tilde{G}_0}$, $w \leftrightarrow (w)$, in the standard way, we have for any $w_1, w_2 \in W_*$:

$$\mathbf{I}_{(w_1)} * \mathbf{I}_{w_2} \cong \bigoplus_{w_3 \in W_*, \delta \in \mathbf{Z}} \mathbf{I}_{(w_3)}[\delta]^{\oplus r_{w_1, w_2, w_3; \delta}}$$

(in $\mathcal{C}_{\tilde{G}_0, \tilde{G}_0}$), where

$$C'_{w_1} C'_{w_2} = \sum_{w_3 \in W_*, \delta \in \mathbf{Z}} r_{w_1, w_2, w_3; \delta} v^{\delta} \pi' C'_{w_3}$$

is a product in the affine Hecke algebra. Here π' is as in A.1 and $r_{w_1, w_2, w_3; \delta}$ are integers. It is known [L1] that

$$r_{w_1, w_2, w_3; \delta} \text{ is zero unless } \delta = 0.$$

(This is also a special case of Proposition A.2.) In view of the above, this result can be reformulated as follows:

(a) If $w_1, w_2 \in W_*$, then $\mathbf{I}_{(w_1)} * \mathbf{I}_{(w_2)}$ is a perverse sheaf on $X_{\tilde{G}_0}$.

(b) If $A, A' \in \mathscr{C}_{\tilde{G}_0, \tilde{G}_0}$ are perverse sheaves, then so is $A * A'$.

The following result is more or less equivalent to (a):

(c) Let $w_1, w_2 \in W_*$. Let Z_1 (resp. Z_2) be the closure of the orbit (w_1) (resp. (w_2)) in $X_{\tilde{G}_0}$. Let Z_3 be the set pairs $(x, x') \in X_{\tilde{G}_0} \times X_{\tilde{G}_0}$ such that $x \in Z_1$ and such that $(gx, gx') \in \{\tilde{G}_0\} \times Z_2$ for some $g \in \tilde{G}$. Let Z_4 be the image of the second projection $pr_2: Z_3 \to X_{\tilde{G}_0}$. Then $pr_2: Z_3 \to Z_4$ is a semismall morphism of irreducible projective varieties.

The proof is based on the same estimates that were used in the proof of (a). Namely, it is enough to consider the case where G is replaced by a group of the same kind over an algebraic closure of a finite field. We must prove certain estimates for the dimension of the fibres of the map pr_2. It is enough to show that the number of points over a finite field F_q of such a fibre is given by a polynomial in q whose degree has a specific bound. But these polynomial can be interpreted, as is well known, in terms of multiplication in the affine Hecke algebra, and the desired bound follows from Proposition A.4.

1.6. Let $\mathscr{M}_0$ be the category of perverse sheaves on $X_{\tilde{G}_0}$ (affine case) that lie in $\mathscr{C}_{\tilde{G}_0, \tilde{G}_0}$. From 1.5(b) we see that $\mathscr{M}_0$ with the "tensor product" $*$ and the obvious associativity constraint is a monoidal category. (It has a unit object given by the sheaf $\mathbf{C}$ supported by the point $\tilde{G}_0 \in X_{\tilde{G}_0}$.)

Let ${}^LG$ be a semisimple group over $\mathbf{C}$ that is a Langlands dual to G. As is well known, the irreducible finite dimensional representations of ${}^LG$ are canonically in bijection with the set W_* (hence, with the simple objects of $\mathscr{M}_0$); let V_w be the irreducible representation corresponding to $w \in W_*$. One of the main results of [L1] is

If $w_1, w_2 \in W_*$, then

$$\mathbf{I}_{(w_1)} * \mathbf{I}_{w_2} \cong \bigoplus_{w_3 \in W_*} \mathbf{I}_{(w_3)}^{\oplus r_{w_1, w_2, w_3}}$$

where r_{w_1, w_2, w_3} is the multiplicity of V_{w_3} in $V_{w_1} \otimes V_{w_2}$.
This can be made more canonical by showing that $\mathscr{M}_0$, with a suitable commutativity restraint, is equivalent as a braided tensor category to the tensor category of finite-dimensional representations of ${}^LG$ (see Ginzburg [Gi]). Note that the "fibre functor" needed to do that is implicit in [L1], where it is shown that, for $w \in W_*$, we have $\dim H^*(X_{\tilde{G}_0}, \mathbf{I}_{(w)}) = \dim V_w$, which forces one to define the fibre functor by $A \mapsto H^*(X_{\tilde{G}_0}, A)$.

This has been extended by Mirkovic and Vilonen who, in a forthcoming paper, show how to reconstruct ${}^L G$ (over any field K) in terms of perverse K-sheaves on $X_{\tilde{G}_0}$, using a very elegant definition of the commutativity restraint, given recently by Drinfeld.

1.7. Let W'_* be the set of all $y \in W$ such that y has maximal length in $W_f y$. We identify $W'_* \Leftrightarrow \mathbf{O}_{\tilde{G}_0, B}$, $w \leftrightarrow \mathbf{w}$, in the standard way. We have the following variant of 1.5(c).

(a) Let $w_1 \in W_*$, $w_2 \in W'_*$. Let Z_1 (resp. Z_2) be the closure of the orbit (w_1) (resp. $\mathbf{w}_2$) in $X_{\tilde{G}_0}$ (resp. X_B). Let Z_3 be the set of pairs $(x, x') \in X_{\tilde{G}_0} \times X_B$ such that $x \in Z_1$ and such that $(gx, gx') \in \{\tilde{G}_0\} \times Z_2$ for some $g \in \tilde{G}$. Let Z_4 be the image of the second projection $pr_2 \colon Z_3 \to X_B$. Then $pr_2 \colon Z_3 \to Z_4$ is a semismall morphism of irreducible projective varieties.

The proof is the same as that of 1.5(c); it uses A.5 instead of A.4. This implies that the question (i) at the end of [Gi, 1.10] has a positive answer: the convolution of a $\tilde{G}_0$-equivariant perverse sheaf on $X_{\tilde{G}_0}$ with a $\tilde{G}_0$-equivariant perverse sheaf on X_B is a $\tilde{G}_0$-equivariant perverse sheaf on X_B.

2. TRUNCATED CONVOLUTION

2.1. The arguments in this section make sense both in the finite and affine case. Recall that, if $D \in \mathscr{C}$, then for any k we have canonical maps

$$ {}^pH^k(D)[-k] \xrightarrow{i} {}^p\tau_{\geqslant k}(D) \xrightarrow{q} {}^p\tau_{\geqslant k+1}(D) $$

such that $qi = 0$ and there exists a (noncanonical) map ${}^p\tau_{\geqslant k+1}(D) \xrightarrow{j} {}^p\tau_{\geqslant k}(D)$ such that $qj = 1$ and such that

$$ {}^p\tau_{\geqslant k+1}(D) \oplus {}^pH^k(D)[-k] \xrightarrow{(j,\, i)} {}^p\tau_{\geqslant k}(D) $$

is an isomorphism whose inverse has q as first component. It follows that if, in addition, $C \in \mathscr{C}$, then we have canonical maps

$$ {}^pH^k(D)[-k] * C \xrightarrow{i'} {}^p\tau_{\geqslant k}(D) * C \xrightarrow{q'} {}^p\tau_{\geqslant k+1}(D) * C $$

such that $q'i' = 0$ and j induces a (noncanonical) map ${}^p\tau_{\geqslant k+1}(D) * C \xrightarrow{j'} {}^p\tau_{\geqslant k}(D) * C$ such that $q'j' = 1$ and such that

$$ {}^p\tau_{\geqslant k+1}(D) * C \oplus {}^pH^k(D)[-k] * C \xrightarrow{(j',\, i')} {}^p\tau_{\geqslant k}(D) * C $$

is an isomorphism whose inverse has q' as first component. Applying ${}^pH^l$, we deduce that we have an induced exact sequence of perverse sheaves

(a) $0 \to {}^pH^{l-k}({}^pH^k(D) * C) \to {}^pH^l({}^p\tau_{\geqslant k}(D) * C)$

$\to {}^pH^l({}^p\tau_{\geqslant k+1}(D) * C) \to 0,$

where the middle maps are induced by i', q'; hence, they are canonical.

2.2. Let $\mathbf{c}$ be a two-sided cell of W (a finite or affine Weyl group) and let $\mathscr{C}_{\mathbf{c}}$ (resp. $\mathscr{C}_{\leqslant \mathbf{c}}$ or $\mathscr{C}_{<\mathbf{c}}$) be the subcategory of $\mathscr{C}$ consisting of complexes which are direct sums of $\mathbf{I}_w[\delta]$ with $w \in \mathbf{c}$ (resp. w in a two-sided cell that is $\leqslant \mathbf{c}$, or $<\mathbf{c}$), where we use the standard partial order (also known as $\leqslant_{LR}$) for two-sided cells [KL].

Let $\mathscr{M}_{\mathbf{c}}$ (resp. $\mathscr{M}_{\leqslant \mathbf{c}}$ or $\mathscr{M}_{<\mathbf{c}}$) be the category of perverse sheaves on $X_{B'}$ that lie in $\mathscr{C}_{\mathbf{c}}$ (resp. $\mathscr{C}_{\leqslant \mathbf{c}}$ or $\mathscr{C}_{<\mathbf{c}}$). From 1.4 and the definitions, we see that

(a) if $A, A' \in \mathscr{C}_{\leqslant \mathbf{c}}$ then $A * A' \in \mathscr{C}_{\leqslant \mathbf{c}}$.

Moreover,

(b) if in addition, either A or A' is in $\mathscr{C}_{<\mathbf{c}}$, then $A * A' \in \mathscr{C}_{<\mathbf{c}}$.

2.3. Let $\mathbf{a}\colon W \to \mathbf{N}$ be the function defined in [L2, I]. Recall that this function is constant on two-sided cells; we denote its value on $\mathbf{c}$ by a. From the definition of the function $\mathbf{a}$, we see that, for any $w_1, w_2 \in W$ and any $w \in \mathbf{c}$, we have $-a \leqslant \delta \leqslant a$ whenever $\mathbf{I}_w[\delta]$ appears as a direct summand of $\mathbf{I}_{w_1} * \mathbf{I}_{w_2}$. This has the following consequence:

(a) If A, A' belong to $\mathscr{M}_{\leqslant \mathbf{c}}$ and $\delta < -a$ or $\delta > a$, then ${}^pH^\delta(A * A') \in \mathscr{M}_{<\mathbf{c}}$.

2.4. For any object $M \in \mathscr{M}_{\leqslant \mathbf{c}}$ we can write canonically $M = M^\heartsuit \oplus M^\clubsuit$, where $M^\heartsuit \in \mathscr{M}_{\mathbf{c}}$ and $M^\clubsuit \in \mathscr{M}_{<\mathbf{c}}$. A morphism $M \to M'$ in $\mathscr{M}_{\leqslant \mathbf{c}}$ is said to be a $\heartsuit$-isomorphism if the induced morphism $M^\heartsuit \to M'^\heartsuit$ is an isomorphism.

Let $A, A' \in \mathscr{M}_{\mathbf{c}}$. We set

$$A \odot A' = {}^pH^a(A * B)^\heartsuit \in \mathscr{M}_{\mathbf{c}},$$

where we have used that ${}^pH^a(A * A') \in \mathscr{M}_{\leqslant \mathbf{c}}$ (see 2.2 (a)).

Let $A, A', A'' \in \mathscr{M}_{\mathbf{c}}$. We set

$$A \odot A' \odot A'' = {}^pH^{2a}(A * A' * A'')^\heartsuit \in \mathscr{M}_{\mathbf{c}},$$

where we have used that ${}^pH^{2a}(A * A' * A'') \in \mathscr{M}_{\leqslant \mathbf{c}}$ (see 2.2 (a)).

2.5. Let $A, A', A'' \in \mathscr{M}_{\mathbf{c}}$. We set $D = A * A'$. We consider the canonical exact sequence 2.1 (a) (in $\mathscr{M}_{\leqslant \mathbf{c}}$) for $l = 2a$ and any k:

(a) $0 \to {}^pH^{2a-k}({}^pH^k(D) * A'') \to {}^pH^{2a}({}^p\tau_{\geqslant k}(D) * A'')$

$\to {}^pH^{2a}({}^p\tau_{\geqslant k+1}(D) * A'') \to 0.$

If $k=a$, then ${}^pH^{2a}({}^p\tau_{\geqslant k+1}(D) * A'') \in \mathscr{M}_{<\mathbf{c}}$. Indeed, ${}^p\tau_{\geqslant a+1}(D) \in \mathscr{C}_{<\mathbf{c}}$ (by 2.3(a)); hence, ${}^p\tau_{\leqslant a+1}(D) * C \in \mathscr{C}_{<\mathbf{c}}$ (by 2.2(b)). Hence, in this case, the canonical injection

$$ {}^pH^a({}^pH^a(D) * A'') \to {}^pH^{2a}({}^p\tau_{\geqslant a}(D) * A'') $$

in (a) is a $\heartsuit$-isomorphism. This induces a canonical isomorphism

(b) $\quad {}^pH^a({}^pH^a(D) * A'')^{\heartsuit} \xrightarrow{\sim} {}^pH^{2a}({}^p\tau_{\geqslant a}(D) * A'')^{\heartsuit}.$

If $k<a$ then $2a-k>a$; hence, ${}^pH^{2a-k}({}^pH^k(D) * A'') \in \mathscr{M}_{<\mathbf{c}}$. (We use 2.3(a) with A, A' replaced by ${}^pH^k(D)$, A''). Hence, in this case the canonical surjection

$$ {}^pH^{2a}({}^p\tau_{\geqslant k}(D) * A'') \to {}^pH^{2a}({}^p\tau_{\geqslant k+1}(D) * A'') $$

in (a) is a $\heartsuit$-isomorphism. This induces a canonical isomorphism,

$$ {}^pH^{2a}({}^p\tau_{\geqslant k}(D) * A'')^{\heartsuit} \xrightarrow{\sim} {}^pH^{2a}({}^p\tau_{\geqslant k+1}(D) * A'')^{\heartsuit}. $$

Composing these isomorphisms for $k=a-1, a-2, \ldots$ and using the fact that $\tau_{\geqslant k}(D)=D$ if $k \ll 0$, we obtain an isomorphism

$$ {}^pH^{2a}(D * A'')^{\heartsuit} \xrightarrow{\sim} {}^pH^{2a}({}^p\tau_{\geqslant a}(D) * A'')^{\heartsuit}. $$

Combining this with the isomorphism (b), we obtain an isomorphism

(c) $\quad {}^pH^{2a}(D * A'')^{\heartsuit} \xrightarrow{\sim} {}^pH^a({}^pH^a(D) * A'')^{\heartsuit}.$

We have ${}^pH^a(D) = {}^pH^a(D)^{\heartsuit} \oplus {}^pH^a(D)^{\clubsuit}$; hence,

(d) $\quad {}^pH^a({}^pH^a(D) * A'') = {}^pH^a({}^pH^a(D)^{\heartsuit} * A'') \oplus {}^pH^a({}^pH^a(D)^{\clubsuit} * A'').$

Here ${}^pH^a({}^pH^a(D)^{\clubsuit} * A'') \in \mathscr{M}_{<\mathbf{c}}$ since ${}^pH^a(D)^{\clubsuit} \in \mathscr{C}_{<\mathbf{c}}$ and $A'' \in \mathscr{C}_{<\mathbf{c}}$. Hence, the direct sum decomposition (d) induces an isomorphism

$$ {}^pH^a({}^pH^a(D) * A'')^{\heartsuit} \xrightarrow{\sim} {}^pH^a({}^pH^a(D)^{\heartsuit} * A'')^{\heartsuit}. $$

Combining this with (c) we obtain an isomorphism

$$ {}^pH^{2a}(A * A' * A'')^{\heartsuit} \xrightarrow{\sim} {}^pH^a({}^pH^a(A * A')^{\heartsuit} * A'')^{\heartsuit}, $$

or, equivalently, an isomorphism $A \odot A' \odot A'' \xrightarrow{\sim} (A \odot A') \odot A''$. An entirely similar argument gives us an isomorphism $A \odot A' \odot A'' \xrightarrow{\sim} A \odot (A' \odot A'')$. Combining this with the previous isomorphism, gives us an isomorphism

(e) $\quad (A \odot A') \odot A'' \xrightarrow{\sim} A \odot (A' \odot A'').$

One can verify that this can be regarded as an *associativity constraint* on the abelian category $\mathscr{M}_{\mathbf{c}}$ in which the tensor product is $A, A' \mapsto A \odot A'$; in other words, it satisfies the *pentagon axiom.*

2.6. Let $J_{\mathbf{c}}$ be the ring attached in [L2, II] to $\mathbf{c}$. Recall that $J_{\mathbf{c}}$ has a $\mathbf{Z}$-basis $(t_w)_{w \in \mathbf{c}}$. From the definition, we see that for $w_1, w_2 \in \mathbf{c}$ we have

$$\mathbf{I}_{w_1} \odot \mathbf{I}_{w_2} \cong \bigoplus_{w_3 \in \mathbf{c}} \mathbf{I}_{w_3}^{\oplus n_{w_1, w_2, w_3}},$$

where $t_{w_1} t_{w_2} = \sum_{w_3 \in \mathbf{c}} n_{w_1, w_2, w_3} t_{w_3}$ is the product in the ring $J_{\mathbf{c}}$.

2.7. For any $A \in \mathscr{C}$ and any $n \in \mathbf{Z}$, let $\mathscr{H}^n_{B'}(A)$ be the stalk at B' of the nth cohomology sheaf of A. Let $\mathscr{D}_{\mathbf{c}}$ be the set of distinguished involutions in $\mathbf{c}$. The definition of $\mathscr{D}_{\mathbf{c}}$ (see [L2, II, 1.3, 1.4]) can be stated in geometric terms as follows:

(a) If $d \in \mathbf{c}$, then $\mathscr{H}^{-n}_{B'}(\mathbf{I}_d) = 0$ if $n > a$.

(b) If $d \in \mathbf{c}$, we have $d \in \mathscr{D}_{\mathbf{c}}$ if and only if $\mathscr{H}^{-a}_{B'}(\mathbf{I}_d) \neq 0$; then automatically $\mathscr{H}^{-a}_{B'}(\mathbf{I}_d)$ is isomorphic to $\mathbf{C}$.

If $d \in \mathscr{D}_{\mathbf{c}}$ then $t_d t_d = t_d$ in the ring $J_{\mathbf{c}}$; hence, by 2.6, there exists an isomorphism $u_d \colon \mathbf{I}_d \xrightarrow{\sim} \mathbf{I}_d \odot \mathbf{I}_d$. We fix such an isomorphism for each $d \in \mathscr{D}_{\mathbf{c}}$. On the other hand, if d, d' are distinct elements of $\mathscr{D}_{\mathbf{c}}$ then $t_d t_{d'} = 0$ in the ring $J_{\mathbf{c}}$; hence, we have $\mathbf{I}_d \odot \mathbf{I}_{d'} = 0$.

Remark 2.8. A choice of an isomorphism u_d is equivalent to a choice of an isomorphism $\mathscr{H}^{-a}_{B'}(\mathbf{I}_d) \xrightarrow{\sim} \mathbf{C}$ (see 2.7(b)). This can be seen as follows. Let $u \in W$. From the definition of $*$ we have $\mathscr{H}^0_{B'}(\mathbf{I}_w * \mathbf{I}_{w^{-1}}) = H^0(\mathscr{B}, \mathbf{I}_w \otimes \mathbf{I}_w)$, where H^0 denotes hypercohomology in degree 0. (More precisely, we should take the hypercohomology of the support of $\mathbf{I}_w \otimes \mathbf{I}_w$ which is a projective variety.) Since $\mathbf{I}_w = \mathscr{D}(\mathbf{I}_w)$ (where $\mathscr{D}$ is Verdier duality) we have

$$H^0(\mathscr{B}, \mathbf{I}_w \otimes \mathbf{I}_w) = H^0(\mathscr{B}, \mathbf{I}_w \otimes \mathscr{D}(\mathbf{I}_w)) = \mathbf{C}$$

(the last equality follows from the proof of [L3, (7.4.2)]). Thus, $\mathscr{H}^0_{B'}(\mathbf{I}_w * \mathbf{I}_{w^{-1}}) = \mathbf{C}$. Now assume that $w \in \mathbf{c}$. We have canonical surjective maps

$$\begin{aligned} \mathbf{C} = \mathscr{H}^0_{B'}(\mathbf{I}_w * \mathbf{I}_{w^{-1}}) &\to \mathscr{H}^0_{B'}({}^p\tau_{\geqslant a}(\mathbf{I}_w * \mathbf{I}_{w^{-1}})) \\ &= \mathscr{H}^0_{B'}({}^pH^a(\mathbf{I}_w * \mathbf{I}_{w^{-1}}[-a])) \\ &= \mathscr{H}^{-a}_{B'}({}^pH^a(\mathbf{I}_w * \mathbf{I}_{w^{-1}})) \to \mathscr{H}^{-a}_{B'}(\mathbf{I}_w \odot \mathbf{I}_{w^{-1}}). \end{aligned}$$

In particular, for $d \in \mathscr{D}_{\mathbf{c}}$, we have a canonical surjective map

(a) $\mathbf{C} \to \mathscr{H}^{-a}_{B'}(\mathbf{I}_d \odot \mathbf{I}_d)$.

But $\mathscr{H}_{B'}^{-a}(\mathbf{I}_d \odot \mathbf{I}_d)$ is isomorphic to $\mathscr{H}_{B'}^{-a}\mathbf{I}_d$, hence to $\mathbf{C}$. Hence (a) is an isomorphism. We see that to specify an isomorphism $\mathbf{I}_d \xrightarrow{\sim} \mathbf{I}_d \odot \mathbf{I}_d$ is the same as to specify the induced isomorphism of one dimensional vector spaces $\mathscr{H}_{B'}^{-a}\mathbf{I}_d \xrightarrow{\sim} \mathscr{H}_{B'}^{-a}(\mathbf{I}_d \odot \mathbf{I}_d) = \mathbf{C}$. Our assertion follows.

2.9. Let $U = \bigoplus_{d \in \mathscr{D}_\mathbf{c}} \mathbf{I}_d \in \mathscr{M}_\mathbf{c}$. We define $u\colon U \xrightarrow{\sim} U \odot U$ or, equivalently,

$$u\colon \bigoplus_{d \in \mathscr{D}_c} \mathbf{I}_d \xrightarrow{\sim} \bigoplus_{d \in \mathscr{D}_\mathbf{c}} (\mathbf{I}_d \odot \mathbf{I}_d)$$

by $u = (u_d)_{d \in \mathscr{D}_\mathbf{c}}$. We show that

(a) the functor $A \mapsto U \odot A$ from $\mathscr{M}_\mathbf{c}$ to $\mathscr{M}_\mathbf{c}$ is an equivalence of categories.

From [L2, II] we know that $\sum_{d \in \mathscr{D}_\mathbf{c}} t_d$ is the unit element of the ring $J_\mathbf{c}$. Using 2.6, it follows that $U \odot A$ is isomorphic to A for any $A \in \mathscr{M}_\mathbf{c}$. In particular, any object of $\mathscr{M}_\mathbf{c}$ is isomorphic to an object of the form $U \otimes A$. Next we must show that the natural map $\mathrm{Hom}(A, A') \to \mathrm{Hom}(U \odot A, U \odot A')$ is an isomorphism for any $A, A' \in \mathscr{M}_\mathbf{c}$. This follows from the fact that A, A' are direct sum of simple objects. Thus, (a) is verified.

Similarly, we see that the functor $A \mapsto A \odot U$ from $\mathscr{M}_\mathbf{c}$ to $\mathscr{M}_\mathbf{c}$ is an equivalence of categories. We see that *$\mathscr{M}_\mathbf{c}$ with the "tensor product" operation $\odot$ and with the associative constraint* 2.5(e) *is a monoidal category with unit object* $(U, u\colon U \xrightarrow{\sim} U \odot U)$.

2.10. Let Γ be a left cell contained in $\mathbf{c}$. Let $\mathscr{M}_\Gamma$ be the subcategory of $\mathscr{M}_\mathbf{c}$ consisting of perverse sheaves on $X_{B'}$ which are direct sums of $\mathbf{I}_w$ with $w \in \Gamma \cap \Gamma^{-1}$. If $A, A' \in \mathscr{M}_\Gamma$, then $A \odot A' \in \mathscr{M}_\Gamma$. Note that $\mathscr{M}_\Gamma$ with the "tensor product" operation $\odot$ and with the associative constraint 2.5(e) is a monoidal category with unit object $(\mathbf{I}_d, u_d\colon \mathbf{I}_d \xrightarrow{\sim} \mathbf{I}_d \odot \mathbf{I}_d)$, where d is the unique element of $\mathscr{D}_\mathbf{c} \cap \Gamma = \mathscr{D}_\mathbf{c} \cap \Gamma^{-1}$.

3. EXAMPLES

3.1. In this section we assume that we are in the affine case. According to [L2, I, 8.5], the set Γ consisting of all $w \in W$ such that w has maximal length in wW_f is a left cell of W. Note that $\Gamma \cap \Gamma^{-1} = W_*$. We have a functor $\mathscr{M}_0 \to \mathscr{M}_\Gamma$ given by $A \mapsto f^*A[v]$, where $f\colon X_{B'} \to X_{\tilde{G}_0}$ is the canonical map with fibres of dimension v. Clearly, this is an equivalence of categories (in fact of monoidal categories where $\mathscr{M}_0$ has tensor product $*$ and $\mathscr{M}_\Gamma$ has tensor product $\odot$).

3.2. According to [L2, IV], there is a natural bijection between the set of two-sided cells of W and the set of unipotent classes of ${}^LG$. Let $u \in {}^LG$ be a unipotent element in the class corresponding to $\mathbf{c}$ and let $H_{\mathbf{c}}$ be the reductive quotient of the centralizer of u in ${}^LG$.

Let Γ be the set of all $w \in \mathbf{c}$ such that w has minimal length in wW_f. Then Γ is a left cell [LX]. It is very likely that the monoidal category $\mathscr{M}_\Gamma$ admits a commutativity constraint and that it is equivalent to the monoidal category of finite dimensional representations of the reductive group $H_{\mathbf{c}}$. (This would be a strengthening of [L2, IV, 10.7].)

Note also that the conjectures in [L2, IV, 10.5, 10.6] (in the affine case) and the analogous conjectures in [L4, 3.15] (in the finite case) can now be extended to conjectures relating the monoidal category $\mathscr{M}_{\mathbf{c}}$ or $\mathscr{M}_\Gamma$ (for any left cell Γ) and a suitable category of equivariant vector bundles with an obvious monoidal structure.

APPENDIX

A.1. Let (W, S) be an affine Weyl group. Let l be the length function on W. We fix a finite subgroup W_f of W of maximal order that is generated by a finite subset of S. Let w_0 be the longest element of W_f; let $\nu = l(w_0)$.

Let v be an indeterminate. Let $\pi = \sum_{w \in W_f} v^{2l(w)}$, $\pi' = v^{-\nu}\pi \in \mathbf{Z}[v, v^{-1}]$. Recall that the Hecke algebra $\mathbf{H}$ associated to W is the $\mathbf{Z}[v, v^{-1}]$-algebra which, as a $\mathbf{Z}[v, v^{-1}]$-module, has basis elements T_w (one for each $w \in W$) and multiplication defined by the rules

$$(T_s + 1)(T_s - v^2) = 0 \qquad \text{for} \quad s \in S;$$

$$T_w T_{w'} = T_{ww'} \qquad \text{if} \quad l(w) + l(w') = l(ww').$$

Let $^- : \mathbf{Z}[v, v^{-1}] \to \mathbf{Z}[v, v^{-1}]$ be the ring involution given by $v^n \mapsto v^{-n}$ for any $n \in \mathbf{Z}$. Let $^- : \mathbf{H} \to \mathbf{H}$ be the ring involution given by $v^n T_w \mapsto v^{-n} T_{w^{-1}}^{-1}$ for $w \in W$, $n \in \mathbf{Z}$.

For $w \in W$ we set $\tilde{T}_w = v^{-l(w)} T_w$. Let $C'_w \in \mathbf{H}$ be the unique element such that $\overline{C'_w} = C'_w$ and such that $C'_w = \sum_{w'} p_{w', w} \tilde{T}_{w'}$, where $p_{w, w} = 1$, $p_{w', w} \in v^{-1}\mathbf{Z}[v^{-1}]$ for $w' < w$, and $p_{w', w} = 0$ if $w' \nleqslant w$ (see [KL]). Here $\leqslant$ is the standard partial order on W.

For $x, y, z \in W$ we define $m_{w, y, z}, m'_{x, y, z} \in \mathbf{Z}[v, v^{-1}]$ by

$$\tilde{T}_x \tilde{T}_y = \sum_{z \in W} m_{x, y, z} \tilde{T}_{z^{-1}};$$

$$C'_x C'_y = \sum_{z \in W} m'_{x, y, z} C'_{z^{-1}}.$$

We recall the following result [L2, I, Theorem 7.2]:

(a) $v^{-\nu}m_{x,y,z} \in \mathbf{Z}[v^{-1}]$.

We show that

(b) $v^{-\nu}m'_{x,y,z} \in \mathbf{Z}[v^{-1}]$.

From definitions we have, for any $x, y, z \in W$:

(c) $\sum_{z'} v^{-\nu}m'_{x,y,z'}p_{z^{-1},z'^{-1}} = \sum_{x',y'} p_{x',x}p_{y',y}v^{-\nu}m_{x',y',z}$.

We argue by descending induction on $l(z)$ with fixed x, y. If $l(z)$ is very large, we have $m'_{x,y,z} = m_{x,y,z} = 0$ and there is nothing to prove. Assume now that $l(z) = l_0$ and that the result is already known when z is replaced by z_1 with $l(z_1) > l_0$. Using (a) and the definition of C'_w we see that the right-hand side of (c) belongs to $\mathbf{Z}[v^{-1}]$. Using the induction hypothesis and the definition of C'_w, we see that the contribution of the terms with $z' > z$ to left-hand side of (c) belongs to $v^{-1}\mathbf{Z}[v^{-1}]$. Hence the left-hand side of (c) is equal to $v^{-\nu}m'_{x,y,z}$ modulo $v^{-1}\mathbf{Z}[v^{-1}]$. This proves (b).

Proposition A.2. *Let $x, y \in W$ be such that x has maximal length in the coset xW_f and y has maximal length in the coset $W_f y$. We have $C'_w C'_y = \pi' \sum_{z \in W} r_z C'_{z^{-1}}$, where $r_z \in \mathbf{Z}$ for all $z \in W$.*

From [KL, (2.3.g)] it follows easily that $C'_x = hC'_{w_0}$, $C'_y = C'_{w_0}h'$ for some $h, h' \in \mathbf{H}$. It is easy to see that $C'_{w_0}C'_{w_0} = \pi' C'_{w_0}$. Hence

$$C'_x C'_y = hC'_{w_0}C'_{w_0}h' = \pi' hC'_{w_0}h' \in \pi'\mathbf{H}.$$

We deduce that, for any $z \in W$, we have $m'_{x,y,z} = r_z\pi'$ where $r_z \in \mathbf{Z}[v, v^{-1}]$. We must show that $r_z \in \mathbf{Z}$. From the definition of $m'_{x,y,z}$ we see that $\overline{m'_{x,y,z}} = m'_{x,y,z}$. From the definition of π' we see that $\overline{\pi'} = \pi'$. It follows that $\overline{r_z} = r_z$. From A.1(b) it follows that $v^{-\nu}r_z\pi' \in \mathbf{Z}[v^{-1}]$; since $v^{-\nu}\pi' \in 1 + v^{-1}\mathbf{Z}[v^{-1}]$, it follows that $r_z \in \mathbf{Z}[v^{-1}]$. Combining this with $\overline{r_z} = r_z$, we see that $r_z \in \mathbf{Z}$. The proposition is proved.

A.3. Let W_* be the set of all $x \in W$ such that x has maximal length in the double coset $W_f x W_f$. For $x \in W_*$ we set $\tau_x = \sum_{x_1 \in W_f x W_f} T_{x_1}$. It is easy to see that, for any $x, y \in W_*$, we have

$$\sum_{x' \in W_*;\, x' < x} \tau_{x'} \sum_{y' \in W_*;\, y' \leqslant y} \tau_{y'} = \sum_{z' \in W_*} \pi g(x, y, z')\,\tau_{z'},$$

where $g(x, y, z') \in \mathbf{Z}[v^2]$.

Proposition A.4. (a) *We have $xw_0y \in W_*$ and $l(xw_0y) = l(xw_0) + l(y) = l(x) + l(w_0y) = l(x) + l(y) - \nu$.*

(b) *If* $z' \not\leqslant xw_0y$, *the* $g(x, y, z') = 0$.

(c) *If* $z' \leqslant xw_0y$, *then* $g(x, y, z')$ *has degree* $\leqslant l(xw_0y) - l(z')$ *(as a polynomial in* v*)*.

Proof. (a) and (b) are well known. We prove (c). From the definitions, we have

$$\pi g(x, y, z') = \sum_{x', y'; x' \leqslant x, y' \leqslant y} v^{l(x') + l(y') - l(z')} m_{x', y', z'^{-1}}.$$

By A.1(a), the general term of the last sum belongs to

$$v^{l(x') + l(y') - l(z')} v^{\nu} Z[v^{-1}] \subset v^{l(x) + l(y) - l(z')} v^{\nu} \mathbf{Z}[v^{-1}];$$

hence $\pi g(x, y, z') \in v^{l(x) + l(y) - l(z')} v^{\nu} \mathbf{Z}[v^{-1}]$. Since $\pi \in v^{2\nu}(1 + v^{-1}\mathbf{Z}[v^{-1}])$, it follows that

$$g(x, y, z') \in v^{l(x) + l(y) - \nu - l(z')} \mathbf{Z}[v^{-1}] = v^{l(xw_0 y) - l(z')} \mathbf{Z}[v^{-1}].$$

The proposition is proved.

A.5. More generally, let W'_* be the set of all $y \in W$ such that y has maximal length in $W_f y$. For $y \in W'_*$ we set $\tau'_y = \sum_{y_1 \in W_f y} T_{y_1}$. If $x \in W_*$ and $y \in W'_*$, we have

$$\sum_{x' \in W_*; x' \leqslant x} \tau_{x'} \sum_{y' \in W'_*; y' \leqslant y} \tau'_{y'} = \sum_{z' \in W'_*} \pi g'(x, y, z') \tau'_{z'},$$

where $g'(x, y, z') \in \mathbf{Z}[v^2]$. The following holds:

(a) We have $xw_0y \in W'_*$ and $l(xw_0y) = l(xw_0) + l(y) = l(x) + l(w_0 y) = l(x) + l(y) - \nu$.

(b) If $z' \not\leqslant xw_0y$, then $g'(w, y, z') = 0$.

(c) If $z' \leqslant xw_0y$, then $g'(x', y, z')$ has degree $\leqslant l(xw_0y) - l(z')$ (as a polynomial in v).

The proof (based on A.1(a)) is the same as that of A.4.

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